

G.C.E. (A.L.) Support Seminar - 2016
Combined Mathematics - Paper I
Answer Guide

Part A

1. Let $f(n) = 4^n + 15n - 1$; $n \in \mathbb{Z}^+$

When $n = 1$, $f(1) = 4 + 15 - 1 = 18 = 9 \times 2$

$\therefore f(1)$ is divisible by 9.

$\therefore$ The statement is true when $n = 1$. (5)

Let us assume that the given expression is divisible by 9 when $n = p$, $p \in \mathbb{Z}^+$

That is, $f(p) = 4^p + 15p - 1 = 9k$; $k \in \mathbb{Z}^+$. (5)

$$f(p+1) = 4^{p+1} + 15(p+1) - 1$$

$$= 4 \cdot 4^p + 15p + 15 - 1$$

$$= 4 [9k - 15p + 1] + 15p + 15 - 1 \quad (5)$$

$$= 4 \times 9k - 45p + 18$$

$$= 9 [4k - 5p + 2]$$

$$= 9\lambda; \lambda = 4k - 5p + 2 \in \mathbb{Z}^+$$

$\therefore f(p+1)$ is divisible by 9.

$\therefore$ The statement is true when $n = p + 1$ (5)

$\therefore$ By the Principle of Mathematical Induction, the given expression is divisible by 9 for all positive integers n . (5)

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- 2.

$$\left(\sqrt{2} + 7^{\frac{1}{5}}\right)^{10} = \sum_{r=0}^{10} {}^{10}C_r \left(2^{\frac{1}{2}}\right)^{10-r} \left(7^{\frac{1}{5}}\right)^r$$

$$T_r = {}^{10}C_{r-1} \left(2\right)^{\frac{11-r}{2}} \left(7\right)^{\frac{r-1}{5}}; \text{ here } 1 \leq r \leq 11. \quad (5)$$

Since 2 and 7 are primes, for a term to be rational $11-r = 2p$ and $r-1 = 5q$;

where $p, q \in \mathbb{Z}^+$. (5)

That is $r \in \{1, 3, 5, 7, 9, 11\} \cap \{1, 6, 11\}$ (5)

$\therefore r = 1$ or 11 . (5)

$$\therefore \text{The sum of the rational terms} = {}^{10}C_0 2^5 + {}^{10}C_{10} 7^2 = 32 + 49 = 81$$

(5)

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3. The number of ways in which a group of 5 can be formed without restriction

$$= {}^{14}C_5$$

$$= 2002 \quad (5)$$

The number of groups with 5 boys

$$= {}^8C_5$$

$$= 56 \quad (5)$$

The number of groups with 5 girls

$$= {}^6C_5$$

$$= 6 \quad (5)$$

∴ The number of ways in which a group of 5 can be formed such that both sexes are represented in the group

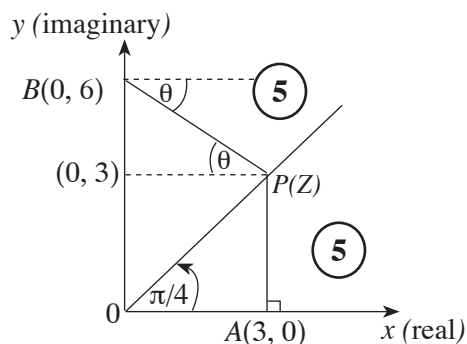
$$= {}^{14}C_5 - ({}^8C_5 + {}^6C_5)$$

$$= 2002 - 56 - 6 \quad (5)$$

$$= 1940 \quad (5)$$

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4.



As indicated in the figure, the point which corresponds to the complex number $Z = Z_0$ such that

$\text{Arg } Z = \frac{\pi}{4}$ and $\text{Arg } (Z - 3) = \frac{\pi}{2}$ is P. (5) According to the figure $\theta = \frac{\pi}{4}$ (5)

$$\therefore \text{Arg } (Z_0 - 6i) = \frac{7\pi}{4} \quad (5)$$

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5.

$$\lim_{x \rightarrow 0} \frac{(1 + kx)^2 - (1 - kx)^2}{\sqrt{1 + k^2x} - \sqrt{1 - k^2x}}$$

$$= \lim_{x \rightarrow 0} \frac{1 + 2kx + k^2x^2 - 1 + 2kx - k^2x^2}{(1 + k^2x) - (1 - k^2x)} \times (\sqrt{1 + k^2x} + \sqrt{1 - k^2x}) \quad (10)$$

$$= \lim_{x \rightarrow 0} \frac{4kx}{2k^2x} (\sqrt{1 + k^2x} + \sqrt{1 - k^2x}) ; k, x \neq 0$$

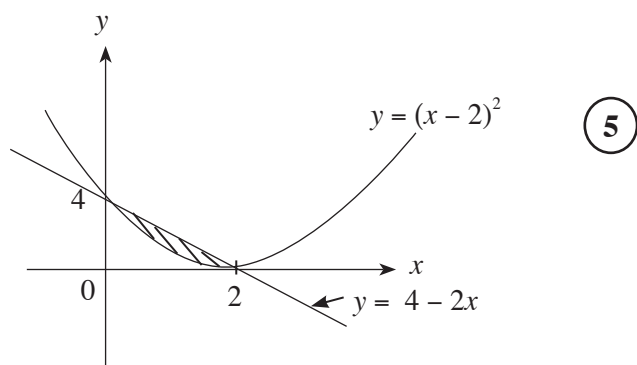
$$= \frac{2}{k} \lim_{x \rightarrow 0} (\sqrt{1 + k^2x} + \sqrt{1 - k^2x}) = \left(\frac{2}{k}\right) \times 2 = \frac{4}{k} \quad (5)$$

$$\frac{4}{k} = 1 \quad (5)$$

$$\therefore k = 4 \quad (5)$$

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6.



$$\begin{aligned}
 \text{Area} &= \int_0^2 \{ (4-2x) - (x-2)^2 \} dx && (5) \\
 &= \int_0^2 (4-2x) dx - \int_0^2 (x-2)^2 dx \\
 &= \left[4x - \frac{2x^2}{2} \right]_0^2 - \left[\frac{(x-2)^3}{3} \right]_0^2 && (5) \\
 &= (8-4) - \left[0 + \frac{8}{3} \right] && (5) \\
 &= 4 - \frac{8}{3} \\
 &= \frac{4}{3} && (5)
 \end{aligned}$$

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7. Differentiating with respect to t ,

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3at^2 - 2t \qquad (5)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = (3at^2 - 2t) \cdot \frac{1}{2t} = \frac{3at - 2}{2} ; t \neq 0 \qquad (5)$$

$$\left(\frac{dy}{dx} \right)_{t=1} = \frac{3a-2}{2}, \quad \left(\frac{dy}{dx} \right)_{t=-1} = \frac{-3a-2}{2} \qquad (5)$$

Since the tangents are perpendicular to each other,

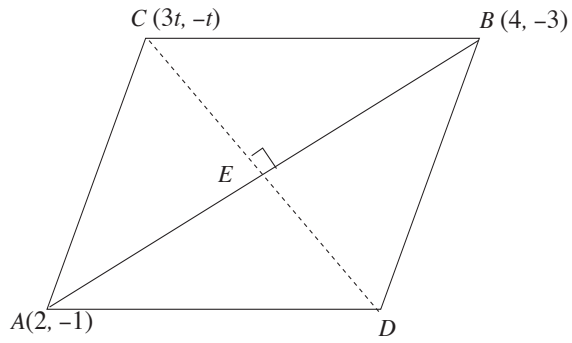
$$\left(\frac{3a-2}{2} \right) \left(\frac{-3a-2}{2} \right) = -1 \qquad (5)$$

$$9a^2 - 4 = 4 \Rightarrow a^2 = 8/9$$

$$\text{Since } a > 0, a = \frac{2\sqrt{2}}{3} \qquad (5)$$

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8.



$$E = (3, -2)$$

Since AB is perpendicular to CE,

$$m_{AB} \times m_{CE} = -1.$$

$$\therefore -1 \times \left(\frac{-2+t}{3-3t} \right) = -1 \quad (5)$$

$$\Rightarrow t = \frac{5}{4} \quad (5)$$

$$\therefore C = \left(\frac{15}{4}, -\frac{5}{4} \right) \quad (5)$$

Let $D = (\bar{x}, \bar{y})$. Then,

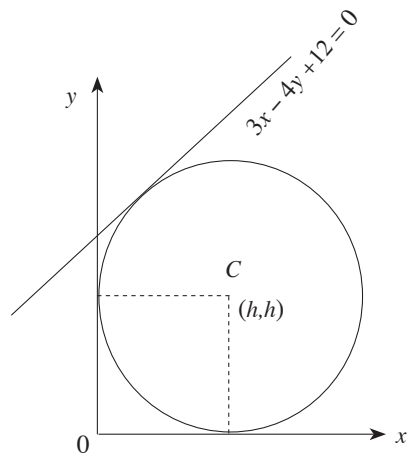
$$\bar{x} = 2 \times 3 - 3t = 6 - 3 \times \frac{5}{4} = \frac{9}{4} \quad (5)$$

$$\bar{y} = 2 \times -2 + t = -4 + \frac{5}{4} = -\frac{11}{4}$$

$$\therefore D = \left(\frac{9}{4}, -\frac{11}{4} \right) \quad (5)$$

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9.



Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a required circle.

Since the circle touches the x and y axes, $C = (h, h)$

(5)

Furthermore, since the line $3x - 4y + 12 = 0$ touches the circle,

$$\left| \frac{3h - 4h + 12}{\sqrt{3^2 + 4^2}} \right| = |h|$$

(5)

$$|-h + 12| = 5|h|$$

(5)

$$\Leftrightarrow (-h + 12) = \pm 5h$$

$$\therefore h = -3 \text{ or } h = 2.$$

(5)

$\therefore$ the equations of the circles are

$$(x - 2)^2 + (y - 2)^2 = 2^2$$

$$(x + 3)^2 + (y + 3)^2 = 3^2$$

(5)

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10. $\cot \alpha - \tan \alpha$

$$= \frac{1}{\tan \alpha} - \tan \alpha$$

$$= \frac{1 - \tan^2 \alpha}{\tan \alpha}$$

$$= \frac{2(1 - \tan^2 \alpha)}{2 \tan \alpha} = \frac{2}{\tan 2\alpha}$$

$$= 2 \cot 2\alpha$$

(5)

$$\cot \alpha - \tan \alpha = 2 \cot 2\alpha \quad \text{--- (1)}$$

$$\therefore \cot 2\alpha - \tan 2\alpha = 2 \cot 4\alpha \quad \text{--- (2)}$$

(5)

$$\cot 4\alpha - \tan 4\alpha = 2 \cot 8\alpha \quad \text{--- (3)}$$

(5)

$$\text{From (1) + 2} \times \text{(2) + 4} \times \text{(3)}$$

$$\cot \alpha - \tan \alpha - 2 \tan 2\alpha - 4 \tan 4\alpha = 8 \cot 8\alpha$$

(10)

$$\cot \alpha = \tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha$$

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Part B

11. (a) $ax^2 + bx + c = 0$

$$a \left[x^2 + \frac{bx}{a} + \frac{c}{a} \right] = 0$$

$$a \left[\left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \right] = 0$$

$$a \left[\left(x + \frac{b}{2a} \right)^2 - \frac{(b^2 - 4ac)}{4a^2} \right] = 0 \quad (10)$$

The condition to have coincident roots is $b^2 - 4ac = 0$. (10)

$$\frac{a}{x+c} + \frac{b}{x-c} = \frac{k}{2x}$$

$$\frac{a(x-c) + b(x+c)}{x^2 - c^2} = \frac{k}{2x}$$

$$x^2[k - 2a - 2b] - 2(bc - ac)x - kc^2 = 0 \quad (10)$$

For coincident roots,

$$4(bc - ac)^2 - 4(k - 2a - 2b)(-kc^2) = 0 \quad (10)$$

That is, $k^2 - 2(a+b)k + (b-a)^2 = 0$ (5)

If k_1 and k_2 are the roots of the above equation, then

$$k_1 + k_2 = 2(a+b) \quad (5) \quad k_1 k_2 = (b-a)^2 \quad (5)$$

$$(k_1 - k_2)^2 = (k_1 + k_2)^2 - 4k_1 k_2 \quad (10)$$

$$= 4(a+b)^2 - 4(b-a)^2$$

$$= 16ab$$

$$\therefore |k_1 - k_2| = 4\sqrt{ab} \quad (10)$$

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(b) $f(x) = (\lambda+1)x^2 + (6-3\lambda)x + (20-12\lambda)$

(i) $f(x)$ is linear when $\lambda = -1$. (5)

(ii) Let the roots be α and $-\alpha$. (5)

$$\text{Then } \alpha + (-\alpha) = -\frac{(6-3\lambda)}{(\lambda+1)} \quad (5)$$

$$\therefore 0 = 6 - 3\lambda. \text{ Hence } \lambda = 2. \quad (5)$$

$$(iii) f(x) = h - b(x - a)^2 = h - b(x^2 - 2ax + a^2) = -bx^2 + 2abx + (h - ba^2)$$

$$f(x) = (\lambda + 1)x^2 + (6 - 3\lambda)x + (20 - 12\lambda) \quad (5)$$

$$\therefore \text{ by comparing the coefficients, } -b = \lambda + 1 \Rightarrow b = -(\lambda + 1) \quad (1) \quad (5)$$

$$2ab = 6 - 3\lambda \Rightarrow a = -\frac{3(2 - \lambda)}{2(\lambda + 1)} \quad (2) \quad (5)$$

$$h - ba^2 = 20 - 12\lambda \Rightarrow h = 4(5 - 3\lambda) - \frac{9}{4} \frac{(2 - \lambda)^2}{(\lambda + 1)} \quad (3) \quad (10)$$

$$\text{Since the maximum value of } f(x) \text{ occurs at } x = 2, \text{ we obtain that } a = 2. \quad (5)$$

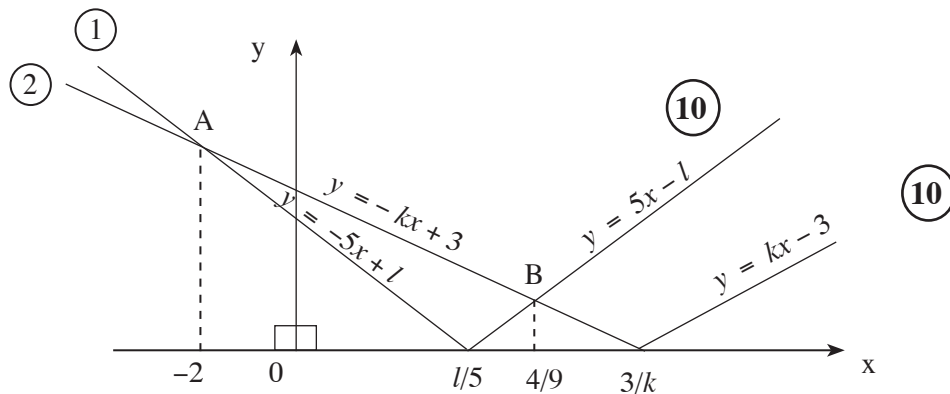
$$(2) \Rightarrow 4(\lambda + 1) = -(6 - 3\lambda) \Rightarrow 4\lambda + 4 = -6 + 3\lambda \Rightarrow \lambda = -10 \quad (5)$$

$$h = 4(5 + 30) - \frac{9}{4} \frac{(2 + 10)^2}{(-10 + 1)} \quad (10)$$

$$\therefore \text{ the maximum value of } f(x) = 176 \quad (10)$$

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12. (a) Since the solution set of the inequality $|l - 5x| < |kx - 3|$ is $\{x \mid -2 < x < 4/9\}$, the two graphs are as illustrated below



$$(1) y = |l - 5x|$$

$$(2) y = |kx - 3|$$

$$\text{For point A: } l + 10 = 2k + 3 \quad (5) \\ l - 2k = -7 \quad (i) \quad (5)$$

$$\text{For point B: } -l + 5 \cdot \frac{4}{9} = -k \cdot \frac{4}{9} + 3 \quad (5) \\ -9l + 4k = 7 \quad (ii) \quad (5)$$

$$\text{by (i) and (ii) } l = 1, \quad (5) \quad k = 4 \quad (5)$$

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$$(b) \quad S_n = \frac{3n}{2n+1} \quad (5)$$

$$\lim_{n \rightarrow \infty} S_n = \frac{3}{2} \quad (5)$$

The limit is finite. (5)

Therefore, the series is convergent. (5)

$$U_r = S_r - S_{r-1} \quad (5)$$

$$= \frac{3r}{2r+1} - \frac{3(r-1)}{2r-1} \quad (5)$$

$$U_r = \frac{3}{4r^2-1} \quad (5)$$

$$\text{Let } S'_n = \sum_{r=1}^n r^2 \frac{3}{4r^2-1} \quad (5)$$

$$= \sum_{r=1}^n \frac{\frac{3}{4}(4r^2-1) + \frac{3}{4}}{(4r^2-1)} \quad (5)$$

$$= \sum_{r=1}^n \frac{3}{4} + \frac{1}{4} \sum_{r=1}^n \frac{3}{4r^2-1} \quad (10)$$

$$= \frac{3n}{4} + \frac{1}{4} S_n \quad (5)$$

$$\quad (5)$$

$$= \frac{3n}{4} + \frac{1}{4} \frac{3n}{(2n+1)} \quad (5)$$

$$= \frac{3n}{4} \left\{ 1 + \frac{1}{2n+1} \right\} \quad (5)$$

$$= \frac{3n(n+1)}{2(2n+1)} \quad (5)$$

$$\text{Therefore } \lim_{n \rightarrow \infty} \sum_{r=1}^n r^2 U_r = \lim_{n \rightarrow \infty} \frac{3n}{4} \left\{ 1 + \frac{1}{2n+1} \right\} \quad (5)$$

$$= \infty \quad (5)$$

The limit is not finite. (5)

$\therefore$ the series is not convergent. (5)

13. (a) $\det A = \begin{vmatrix} 3 & p \\ -2 & -3 \end{vmatrix} = -9 + 2p$ (5)

A^{-1} exists only if $\det A \neq 0$.

That is, if $p \neq 9/2$ (5)

$$A^{-1} = \frac{1}{(2p-9)} \begin{bmatrix} -3 & -p \\ 2 & 3 \end{bmatrix} \quad (5)$$

$$A^{-1} = A$$

$$\frac{1}{(2p-9)} \begin{bmatrix} -3 & -p \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & p \\ -2 & -3 \end{bmatrix} \quad (5)$$

Comparing the corresponding elements,

$$-\frac{3}{2p-9} = 3 \quad \frac{-p}{2p-9} = p \quad (5)$$

$$\frac{2}{2p-9} = -2, \quad \frac{3}{2p-9} = -3 \quad (5)$$

$$(5)$$

$$(5)$$

$$\Rightarrow 2p-9 = -1 \text{ and } p[1+2p-9] = 0$$

Since $p \neq 0$ we have that $p = 4$ (5)

Therefore, $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$

$$A^{-1} = A$$

$$\Rightarrow AA^{-1} = A \cdot A = A^2 \quad (5)$$

$$\therefore I = A^2$$

$$\Rightarrow 0 = A^2 - I$$

$$\Rightarrow 0 = (A - I)(A + I); I^2 = I \quad (5)$$

This is of the form, $0 = BC$,

where $B = A - I = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (5)

$$= \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix} = 2 \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$(5)$$

and

$$C = A + I = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(5)

$$= \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} = 2 \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \quad (5)$$

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(b) (i) Let $Z = x + iy$, where $x, y \in \mathbb{R}$ (5)

$$Z \bar{Z} = (x + iy)(x - iy)$$

$$= x^2 + y^2 \quad (5)$$

$$= \left(\sqrt{x^2 + y^2}\right)^2 = |Z|^2$$

$$\therefore Z \bar{Z} = |Z|^2.$$

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(ii) Let $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

$$Z_1 Z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

$$\therefore \overline{Z_1 Z_2} = (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2)$$

$$= x_1(x_2 - iy_2) - iy_1(-iy_2 + x_2)$$

$$= (x_1 - iy_1)(x_2 - iy_2)$$

$$\overline{Z_1 Z_2} = \bar{Z}_1 \bar{Z}_2$$

(iii) $\left| \frac{\bar{Z}_1 - 2\bar{Z}_2}{2 - Z_1 \bar{Z}_2} \right| = 1$

$$\Rightarrow |\bar{Z}_1 - 2\bar{Z}_2| = |2 - Z_1 \bar{Z}_2| \quad (5)$$

$$\Rightarrow |\bar{Z}_1 - 2\bar{Z}_2|^2 = |2 - Z_1 \bar{Z}_2|^2 \quad (5)$$

$$\Rightarrow (\bar{Z}_1 - 2\bar{Z}_2) \overline{(\bar{Z}_1 - 2\bar{Z}_2)} = (2 - Z_1 \bar{Z}_2) \overline{(2 - Z_1 \bar{Z}_2)} \quad (5)$$

$$\Rightarrow (\bar{Z}_1 - 2\bar{Z}_2)(Z_1 - 2Z_2) = (2 - Z_1 \bar{Z}_2)(2 - \bar{Z}_1 Z_2) \quad (5)$$

$$Z_1 \bar{Z}_1 - 2 \bar{Z}_1 Z_2 - 2 \bar{Z}_2 Z_1 + 4 Z_2 \bar{Z}_2 = 4 - 2 \bar{Z}_1 Z_2 - 2 Z_1 \bar{Z}_2 + Z_1 \bar{Z}_1 Z_2 \bar{Z}_2 \quad (5)$$

$$|Z_1|^2 + 4|Z_2|^2 = 4 + |Z_1|^2 |Z_2|^2$$

$$|Z_1|^2 + 4|Z_2|^2 - |Z_1|^2 \cdot |Z_2|^2 - 4 = 0$$

$$|Z_1|^2 (1 - |Z_2|^2) - 4 (1 - |Z_2|^2) = 0$$

$$(1 - |Z_2|^2) (|Z_1|^2 - 4) = 0 \quad (5)$$

$$\text{Since } |Z_2| \neq 1, \quad |Z_1|^2 - 4 = 0$$

$$\therefore |Z_1|^2 = 4$$

$$\text{Since } |Z_1| > 0, \quad |Z_1| = 2$$

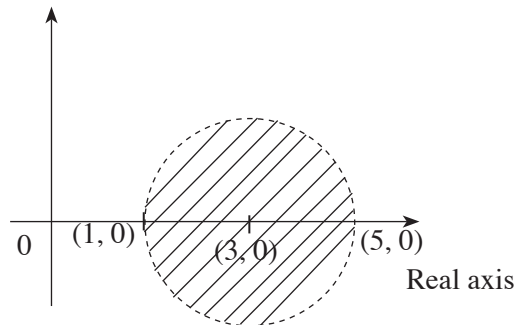
(5)

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(c)

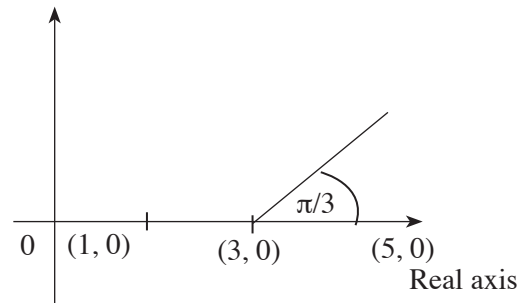
$$|Z - 3| < 2$$

Imaginary axis



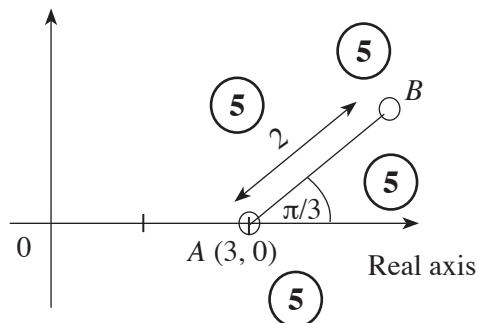
$$\text{Arg}(Z - 3) = \frac{\pi}{3}$$

Imaginary axis



$\therefore$ The locus of the point P which represents the complex number Z such that $|Z - 3| < 2$ and $\text{Arg}(Z - 3) = \frac{\pi}{3}$;

Imaginary axis



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14. (a) $y = (\sin x)^x \quad 0 \leq x \leq \frac{\pi}{2}$

$$\ln y = x \ln |\sin x| \quad (10)$$

$$\frac{1}{y} \frac{dy}{dx} = \ln |\sin x| + x \cot x \quad (10)$$

$$\therefore \frac{dy}{dx} = [x \cot x + \ln (\sin x)] (\sin x)^x \quad (5)$$

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(b) Volume of the tank $= \pi x^2 y + \frac{2}{3} \pi x^3$

$$\therefore \pi x^2 y + \frac{2}{3} \pi x^3 = 45\pi \quad (5)$$

$$\therefore 45 = x^2 \left(y + \frac{2}{3} x \right)$$

$$y = \frac{45}{x^2} - \frac{2}{3} x \quad (5)$$

Surface area of the tank

$$A = 2\pi x^2 + \pi x^2 + 2\pi xy \quad (10)$$

$$A = 3\pi x^2 + 2\pi xy$$

$$A = 3\pi x^2 + 2\pi x \left(\frac{45}{x^2} - \frac{2}{3} x \right)$$

$$A = 3\pi x^2 + \frac{90\pi}{x} - \frac{4\pi}{3} x^2$$

$$A = \frac{5\pi}{3} x^2 + \frac{90\pi}{x} \quad (5)$$

$$\frac{dA}{dx} = \frac{10\pi x}{3} - \frac{90\pi}{x^2} \quad (5)$$

$$= \frac{10\pi (x^3 - 27)}{3x^2}$$

$$= \frac{10\pi}{3x^2} (x - 3)(x^2 + 3x + 9) \quad (5)$$

$$\frac{dA}{dx} = 0 \text{ when } x = 3. \quad (5)$$

x	$0 < x < 3$	$3 < x$
$\frac{dA}{dx}$	< 0	> 0

(5)

$\therefore$ The surface area is minimum when $x = 3$. (5)

$$y = \frac{45}{9} - \frac{6}{3}$$

$$= 3 \quad (5)$$

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$$(c) \quad f(x) = \frac{a}{(x-1)^2} + \frac{b}{(x+1)}$$

Since $f(0) = 2$,

$$a + b = 2 \quad \text{--- (1)} \quad \text{(5)}$$

$$f'(x) = -\frac{2a}{(x-1)^3} - \frac{b}{(x+1)^2} \quad \text{(5)}$$

Since $f'(0) = 0$,

$$2a - b = 0 \quad \text{--- (2)} \quad \text{(5)}$$

$$\text{Form (1) and (2), } a = \frac{2}{3}, \quad b = \frac{4}{3} \quad \text{(5)}$$

$$f'(x) = -\frac{4}{3} \frac{1}{(x-1)^3} - \frac{4}{3(x+1)^2} \quad \text{(5)}$$

$$= -\frac{4}{3} \left\{ \frac{(x+1)^2 + (x-1)^3}{(x-1)^3 (x+1)^2} \right\}$$

$$= -\frac{4}{3} \left[\frac{x^3 - 2x^2 + 5x}{(x-1)^3 (x+1)^2} \right]$$

$$= -\frac{4x}{3} \left[\frac{x^2 - 2x + 5}{(x-1)^3 (x+1)^2} \right]$$

$$= -\frac{4x}{3} \left[\frac{(x-1)^2 + 4}{(x-1)^3 (x+1)^2} \right]$$

Since $(x-1)^2 + 4 > 0$ for all $x \in \mathbb{R}$, we have that $f'(x) = 0$ if and only if $x = 0$. (5)

x	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f'(x)$	< 0	< 0	> 0	< 0
	f decreases	f decreases	f increases	f decreases

(10)

The function f has a relative minimum at $x = 0$. (5)

Then $f(0) = 2$.

When $x \longrightarrow \pm \infty$ we have that $f(x) \longrightarrow 0$.

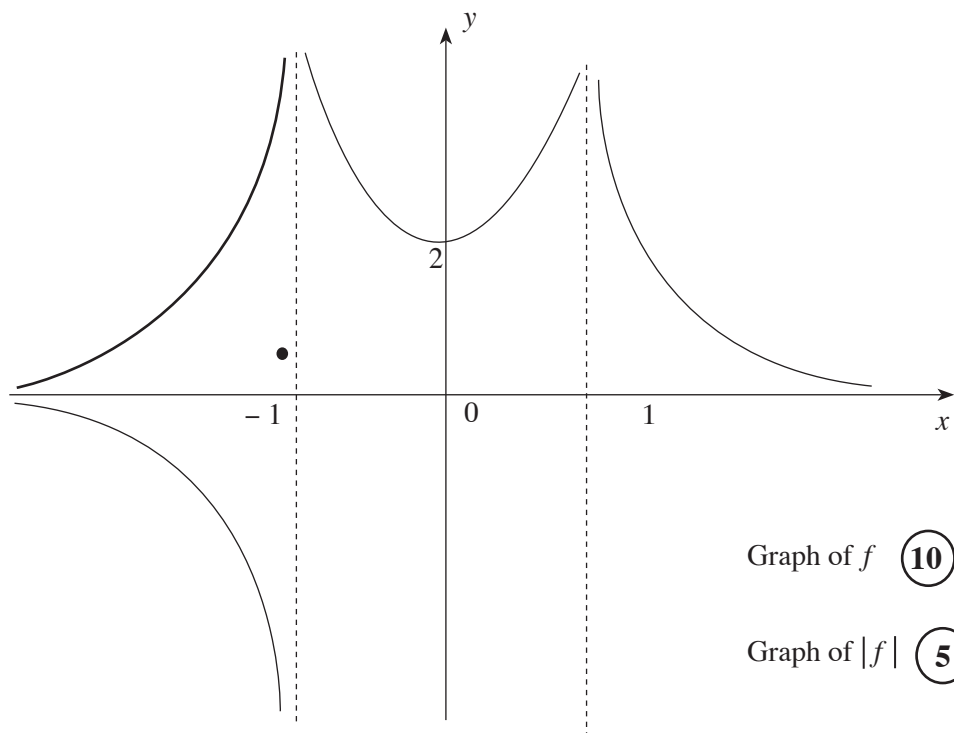
$$x \longrightarrow -1^-, f(x) \longrightarrow -\infty$$

$$x \longrightarrow -1^+, f(x) \longrightarrow +\infty$$

$$x \longrightarrow 1^-, f(x) \longrightarrow +\infty$$

$$x \longrightarrow 1^+, f(x) \longrightarrow +\infty$$

(10)



Graph of f (10)

Graph of $|f|$ (5)

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15. (a) $I = \int_0^1 \frac{dx}{(2+x)^{1/2} (2-x)^{3/2}} = \int_0^1 \frac{dx}{(4-x^2)^{1/2} (2-x)}$

By substituting $x = 2 \sin \theta$, (5)

$dx = 2 \cos \theta \, d\theta$ (5)

$x = 0, \sin \theta = 0$

$\theta = 0$

$x = 1, \sin \theta = \frac{1}{2}$
 $\theta = \frac{\pi}{6}$ (5)

$I = \int_0^{\pi/6} \frac{2 \cos \theta}{(4 - 4 \sin^2 \theta)^{1/2} (2 - 2 \sin \theta)} \, d\theta$ (5)

$= \int_0^{\pi/6} \frac{2 \cos \theta}{2 \cos \theta \cdot 2(1 - \sin \theta)} \, d\theta$

$= \frac{1}{2} \int_0^{\pi/6} \frac{1 + \sin \theta}{\cos^2 \theta} \, d\theta = \frac{1}{2} \int_0^{\pi/6} \sec^2 \theta \, d\theta + \frac{1}{2} \int_0^{\pi/6} \sec \theta \tan \theta \, d\theta$ (5)

$= \frac{1}{2} [\tan \theta]_0^{\pi/6} + \frac{1}{2} [\sec \theta]_0^{\pi/6}$ (5)

$$= \frac{1}{2} \left[\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - 1 \right] = \frac{\sqrt{3}-1}{2}$$

50

(b) $G(x) = \frac{A}{x+2} + \frac{Bx+C}{x^2+8}$

$$1 = A(x^2+8) + (Bx+C)(x+2)$$

$$\left. \begin{array}{l} \text{Coefficient of } x^2 : 0 = A + B \Rightarrow A = -B \\ \text{Coefficient of } x : 0 = 2B + C \Rightarrow C = -2B \end{array} \right\}$$

$$\text{Constant: } 1 = 8A + 2C$$

$$1 = -8B - 4B \Rightarrow 12B = -1$$

$$\Rightarrow B = -\frac{1}{12}$$

$$A = \frac{1}{12}, \quad C = \frac{1}{6}$$

$$g(x) = \int \frac{1}{(x+2)(x^2+8)} dx$$

$$g(x) = \frac{1}{12} \int \frac{1}{(x+2)} dx - \frac{1}{12} \int \frac{x}{(x^2+8)} dx + \frac{1}{6} \int \frac{1}{(x^2+8)} dx$$

$$= \frac{1}{12} \ln|x+2| - \frac{1}{24} \ln(x^2+8) + \frac{1}{6} \tan^{-1} \left(\frac{x}{2\sqrt{2}} \right) + C$$

$$= \frac{1}{24} \ln \left[\frac{(x+2)^2}{x^2+8} \right] + \frac{1}{6} \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x}{2\sqrt{2}} \right) + C$$

$$= \frac{1}{24} \ln \left[\frac{(x+2)^2}{x^2+8} \right] + \frac{1}{12\sqrt{2}} \tan^{-1} \left(\frac{x}{2\sqrt{2}} \right) + C$$

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(c) $I_n = \int x^n \sin x dx$

$$= \int -x^n \frac{d}{dx} (\cos x)$$

$$= -x^n \cos x + \int (\cos x) nx^{n-1} dx$$

$$= -x^n \cos x + n \int x^{n-1} \frac{d}{dx} (\sin x) \quad (5)$$

$$= -x^n \cos x + n \left\{ x^{n-1} \sin x - \int \sin x (n-1) x^{n-2} dx \right\} \quad (10)$$

$$= -x^n \cos x + nx^{n-1} \sin x - n(n-1) I_{n-2} \quad (5)$$

$$I_n + n(n-1) I_{n-2} = x^{n-1} [n \sin x - x \cos x] \quad (5)$$

40

16. (a)

Let $P(\bar{x}, \bar{y})$ be an arbitrary point on any one of the angle bisectors.

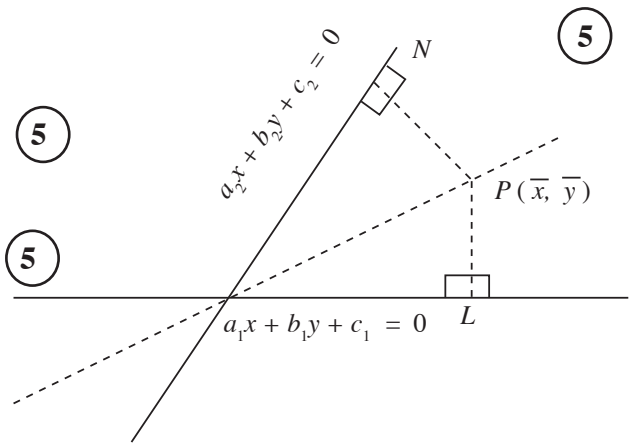
$$PL = PN \quad (5)$$

$$\frac{|a_1 \bar{x} + b_1 \bar{y} + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2 \bar{x} + b_2 \bar{y} + c_2|}{\sqrt{a_2^2 + b_2^2}} \quad (5)$$

$$\therefore \frac{a_1 \bar{x} + b_1 \bar{y} + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 \bar{x} + b_2 \bar{y} + c_2}{\sqrt{a_2^2 + b_2^2}} \quad (5)$$

By replacing $\bar{x}$ by x , and $\bar{y}$ by y (5)

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad (5)$$



The equations of the angle bisectors;

$$\frac{4x + y + 3}{\sqrt{4^2 + 1^2}} = \pm \frac{x + 4y - 3}{\sqrt{1^2 + 4^2}} \quad (5)$$

$$+ : 3x - 3y + 6 = 0 \Rightarrow x - y + 2 = 0 \quad (5)$$

$$- : 5x + 5y = 0 \Rightarrow x + y = 0$$

By solving $x + y = 0$ and $x - y + 2 = 0$ we obtain

$$x = -1, \quad y = 1 \quad (5)$$

Let $A = (-1, 1)$

$B = (0, 2)$ lies on the line given by $x - y + 2 = 0$. (5)

Let $P = (x, y)$ be a point on the line given by $x + y = 0$

Since PA is perpendicular to PB ,

$$\left(\frac{y-1}{x+1}\right) \times 1 = -1 \quad (5)$$

$$\frac{y-1}{-1} = \frac{x+1}{1} = t; \quad t \text{ is a parameter.} \quad (5)$$

$$\therefore x = -1 + t, \quad y = 1 - t$$

Let T be the value of t corresponding to the point D which lies on $x + y = 0$ and is such that $AD = AB$.

$$\text{Then } D = (-1 + T, 1 - T) \quad (5)$$

$$AD^2 = AB^2 \Rightarrow T^2 + T^2 = 1^2 + 1^2 = 2 \quad (5)$$

$$T = \pm 1 \quad (5)$$

$$\therefore D = (0, 0) \text{ or } (-2, +2) \quad (5) \quad (5)$$

When $D = (0, 0)$ the equation of the side CD is

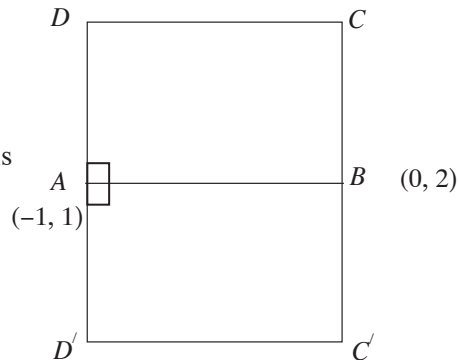
$$x - y = 0 \quad (5)$$

When $D' = (-2, +2)$ the equation of the side $C'D'$ is

$$x - y + 4 = 0 \quad (5)$$

The equation of the side BC and the side BC' is

$$x + y - 2 = 0 \quad (5)$$



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$$(b) \quad S^1 = x^2 + y^2 - 2x + 4y - 3 = 0$$

Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$. Here g, f, c are constants. (5)

Since $S = 0$ is bisected by $S^1 = 0$, the centre of $S = 0$ lies on the line $S^1 - S = 0$ given by (5)

$$-2x(g+1) - 2y(f-2) - 3 - c = 0 \quad (5)$$

$$\therefore 2(g)(g+1) + 2(f)(f-2) - c - 3 = 0 \quad (1) \quad (5)$$

Since the circle $S = 0$ passes through the point $(1, 1)$,

$$1^2 + 1^2 + 2g + 2f + c = 0$$

$$\therefore c = -2g - 2f - 2 \quad (2) \quad (5)$$

From (1) and (2),

$$2g^2 + 2g + 2f^2 - 4f - (-2g - 2f - 2) - 3 = 0 \quad (5)$$

$$2g^2 + 2f^2 + 4g - 2f - 1 = 0 \quad (5)$$

$$2(-g)^2 + 2(-f)^2 - 4(-g) + 2(-f) - 1 = 0$$

$$\therefore \text{ the point } (-g, -f) \text{ lies on the circle } 2x^2 + 2y^2 - 4x + 2y - 1 = 0 \quad (5)$$

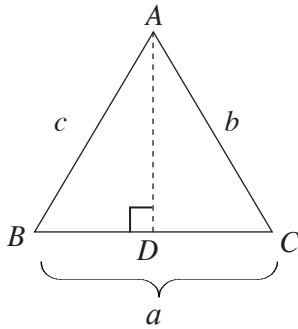
The centre of this circle is $(1, -\frac{1}{2})$ and its radius is (5)

$$r = \sqrt{1^2 + \frac{1}{4} + \frac{1}{2}} = \sqrt{\frac{7}{4}}$$

$$= \frac{\sqrt{7}}{2} \quad (5)$$

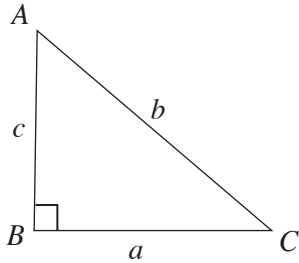
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17. (a)



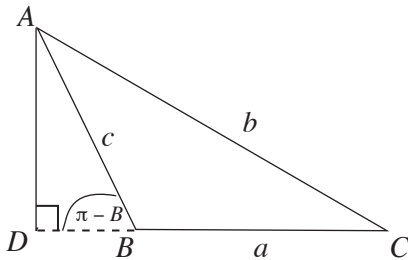
$$\begin{aligned} a &= BC = BD + DC \\ a &= c \cos B + b \cos C \end{aligned}$$

(5)



$$\begin{aligned} a &= b \cos C + 0 \\ &= b \cos C + c \cos 90^\circ \\ &= b \cos C + c \cos B \end{aligned}$$

(5)



$$\begin{aligned} a &= BC = CD - BD \\ &= b \cos C - c \cos (\pi - B) \\ &= b \cos C + c \cos B \end{aligned}$$

(5)

Similarly, $b = a \cos C + c \cos A$

$$a \cos C = b - c \cos A$$

$$a^2 \cos^2 C = b^2 - 2bc \cos A + c^2 \cos^2 A$$

$$a^2 - a^2 \sin^2 C = b^2 + c^2 - 2bc \cos A - c^2 \sin^2 A$$

(10)

$$\underbrace{a^2 + c^2 \sin^2 A - a^2 \sin^2 C}_{= 0} = b^2 + c^2 - 2bc \cos A ; \text{ since } \frac{a}{\sin A} = \frac{c}{\sin C} \quad (5)$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

a, b, c are in an arithmetic progression. Therefore,

$$a + c = 2b \quad (5)$$

$$b \cos C + c \cos B + a \cos B + b \cos A = 2b \quad (5)$$

$$\cos A + \cos C + 2 \cos B = 2$$

$$2 \cos \left(\frac{A+C}{2} \right) \cos \left(\frac{A-C}{2} \right) = 2(1 - \cos B)$$

$$(5)$$

$$2 \cos \left(\frac{\pi}{2} - \frac{B}{2} \right) \cos \left(\frac{A-C}{2} \right) = 4 \sin^2 \frac{B}{2}$$

$$\cos \left(\frac{A-C}{2} \right) = 2 \sin \frac{B}{2} \quad (5)$$

50

(b) $0 < x, y < \frac{\pi}{2}$

$$\therefore 0 < \frac{\pi}{2} - y < \frac{\pi}{2} \quad (5)$$

$$\sin x > \cos y = \sin \left(\frac{\pi}{2} - y \right) \quad (5)$$

$$\sin x > \sin \left(\frac{\pi}{2} - y \right)$$

As the angle increases in the domain $(0, \frac{\pi}{2})$, the sine value also increases.

Therefore, (10)

$$\therefore x > \frac{\pi}{2} - y \quad (5)$$

$$x + y > \frac{\pi}{2}$$

25

(c) $f(x) = 3 \cos^2 x + 8 \sin x \cos x - 3 \sin^2 x$

$$= 3 \cos 2x + 4 \sin 2x \quad (5)$$

$$(5)$$

$$= 5 \left(\frac{3}{5} \cos 2x + \frac{4}{5} \sin 2x \right) \quad (5)$$

$$= 5(\sin \alpha \cos 2x + \cos \alpha \sin 2x)$$

$$= 5 \sin(2x + \alpha)$$

$$= A \sin(2x + \alpha) \quad (5)$$

Here $A = 5$, and α is an acute angle such that $\tan \alpha = \frac{3}{4}$. (5)

$$f(x) = \frac{5}{2}$$

$$5 \sin(2x + \alpha) = \frac{5}{2}$$

$$\sin(2x + \alpha) = \frac{1}{2} = \sin \frac{\pi}{6} \quad (5)$$

$$2x + \alpha = n\pi + (-1)^n \frac{\pi}{6} \quad (5)$$

$$x = \frac{n\pi}{2} - \frac{\alpha}{2} + (-1)^n \frac{\pi}{12}, \text{ here } n \in \mathbb{Z}$$

$$(5)$$

$$f(x) = 5 \sin(2x + \alpha)$$

$$\text{Maximum } f(x) = 5 ; x = \frac{\pi}{4} - \frac{\alpha}{2}$$

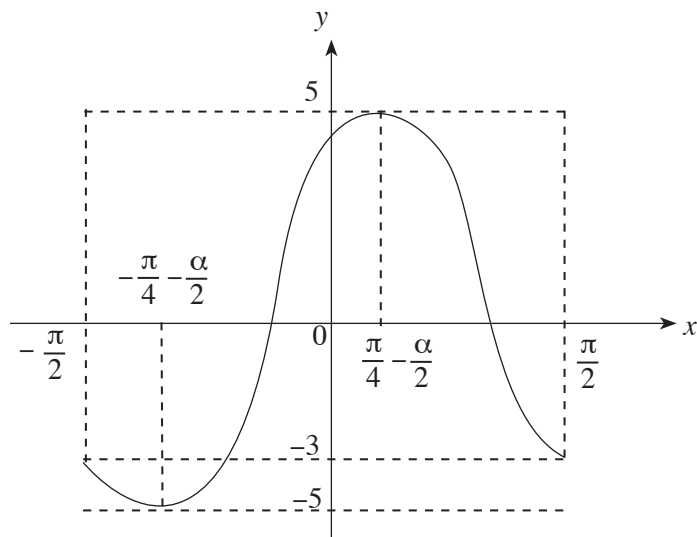
$$(5)$$

$$(5)$$

$$\text{Minimum } f(x) = -5 ; x = -\frac{\pi}{4} - \frac{\alpha}{2} \quad (\text{Since } \alpha < \frac{\pi}{4})$$

$$(5)$$

$$(5)$$



$$(15)$$

G.C.E. (A.L.) Support Seminar - 2016
Combined Mathematics - Paper II
Answer Guide

Part A

1. Applying $v^2 = u^2 + 2as$ to $m \downarrow$

$$v^2 = 2gh$$

$$\therefore v = \sqrt{2gh}$$

(5)

Applying $I = \Delta(mv)$;

To P and $m \downarrow$

$$-J = (2m + m)v_1 - mv - 2m \times 0$$

(5)

$$\therefore -J = 3mv_1 - mv \quad \text{--- (1)}$$

(1)

To $Q \uparrow$

$$J = 2mv_1 - 0$$

(2)

(5)

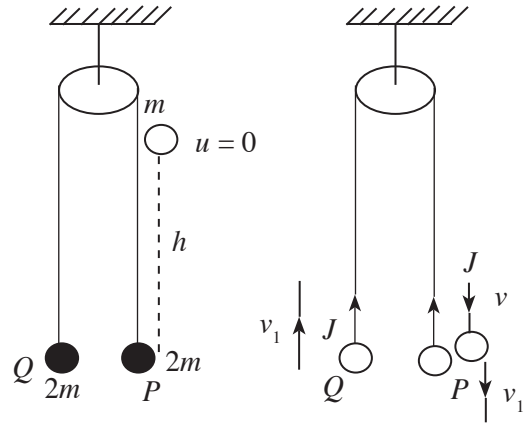
From (1) and (2),

$$v_1 = \frac{v}{5} = \frac{\sqrt{2gh}}{5}$$

(5)

$$J = \frac{2m}{5} \sqrt{2gh}$$

(5)



25

2. The volume of water that is ejected in a second

$$= 8 (0.005) \text{ m}^3$$

$$= 0.040 \text{ m}^3$$

(5)

The mass of water that is ejected in a second

$$= 10^3 \times 0.040 \text{ kg}$$

$$= 40 \text{ kg}$$

(5)

Work done by the pump in a second

$$= mgh + \frac{1}{2} mv^2$$

$$= (40 \times 10 \times 4) + \frac{1}{2} \times 40 \times 8^2$$

$$= 2880 \text{ js}^{-1}$$

(5)

(5)

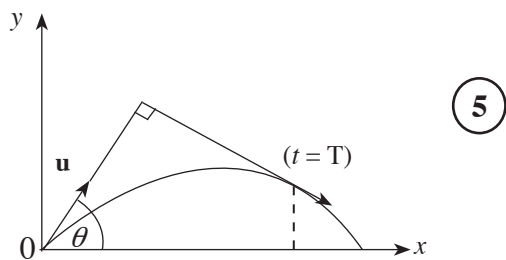
$\therefore$ the power of the pump

$$= 2880 \text{ W}$$

(5)

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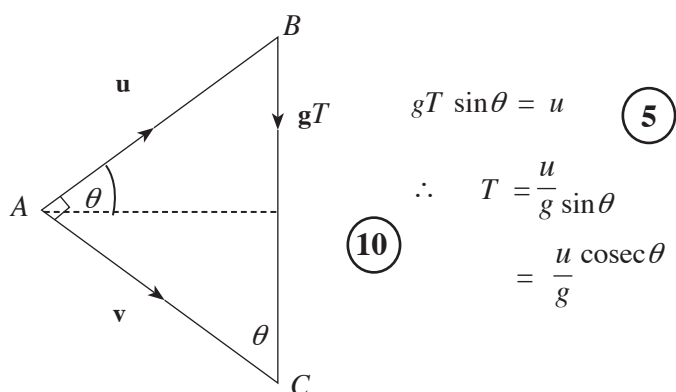
3.



When $t = T$

$$\mathbf{v} = \mathbf{u} + \mathbf{g}T \quad (5)$$

$$\vec{AC} = \vec{AB} + \vec{BC}$$



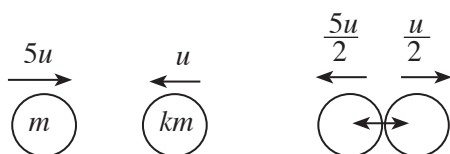
$$gT \sin \theta = u \quad (5)$$

$$\therefore T = \frac{u}{g \sin \theta}$$

$$= \frac{u \operatorname{cosec} \theta}{g}$$

25

4.



By apply the law of conservation of momentum to the system

$$\longrightarrow 5mu - kmu = \frac{km u}{2} - \frac{5mu}{2}$$

$$10 - 2k = k - 5$$

$$\therefore k = 5$$

From Newton's law of restitution

$$\frac{u}{2} + \frac{5u}{2} = e(u + 5u) \quad (5)$$

$$3u = 6ue$$

$$\frac{1}{2} = e \quad (5)$$

$$I = \Delta(mv)$$

$$\longrightarrow -I = -m \cdot \frac{5u}{2} - m \cdot 5u$$

$$I = \frac{15mu}{2} \quad (5)$$

25

5. Since $\underline{a} \perp \underline{b}$, we have $\underline{a} \cdot \underline{b} = 0$ (5)

$\therefore (2\mathbf{i} + 3\mathbf{j}) \cdot (\lambda\mathbf{i} + \mu\mathbf{j}) = 0$

$2\lambda + 3\mu = 0$ (1) (5)

Since $|\underline{b}| = 1$, we have $\lambda^2 + \mu^2 = 1$ (2) (5)

From (1) and (2), $\mu = \pm \frac{2}{\sqrt{13}}$

Since $\mu > 0$, $\mu = \frac{2}{\sqrt{13}}$ (5)

From (1), $\lambda = -\frac{3}{\sqrt{13}}$ (5)

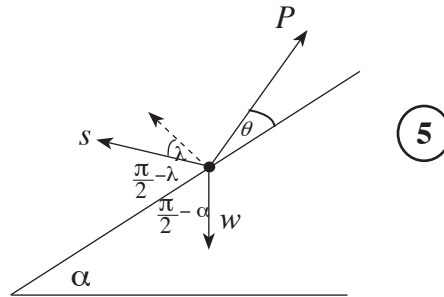
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6. The object is in limiting equilibrium.

By Lami's Theorem,

$\frac{P}{\sin[\pi - (\alpha + \lambda)]} = \frac{w}{\sin[\frac{\pi}{2} - (\theta - \lambda)]}$ (5)

$P = \frac{w \sin(\lambda + \alpha)}{\cos(\theta - \lambda)}$ (5)



For P to be minimum, $(\theta - \lambda)$ should be maximum.

That is, $\theta = \lambda$ (5)

$\therefore P(\text{minimum}) = w \sin(\lambda + \alpha)$ (5)

25

7. Let $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$.

(A) $\longrightarrow$ 1st (First) (B) $\longrightarrow$ 2nd (Second)

(i) $X = (A \cap B') \cup (A' \cap B)$ (5)

But $(A \cap B') \cap (A' \cap B) = \phi$

$\therefore P(X) = P(A \cap B') + P(A' \cap B)$ ($\because$ By Axiom III)

$= P(A) P(B') + P(A') P(B)$ (5) (Since the events are independent)

$= \frac{1}{3} \times \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{3}\right) \cdot \frac{1}{4}$

$= \frac{1}{4} + \frac{2}{3} \times \frac{1}{4} = \left(\frac{1}{4} \times \frac{5}{3}\right) = \frac{5}{12}$ (5)

(ii) $P(A|X) = \frac{P(A \cap X)}{P(X)} = \frac{P(A) P(B')}{P(X)}$ (5)

$= \frac{\frac{1}{3} \times \frac{3}{4}}{\frac{5}{12}} = \frac{3}{5}$ (5)

25

$$\begin{aligned}
 8. \quad P(A \cap B') &= 0.2, \quad P(A' \cap B) &= 0.1 & (5) \\
 P(A' \cap B') &= P(A \cup B)' &= 0.6 \\
 1 - P(A \cup B) &= 0.6 \\
 P(A \cup B) &= 0.4 & (5) \\
 P(A \cup B) - P(A \cap B) &= 0.2 + 0.1 \\
 \therefore P(A \cap B) &= 0.4 - 0.3 = 0.1 & (5) \\
 P(A' \cap B) &= P(B) - P(A \cap B) \\
 0.1 + 0.1 &= P(B) & (5) \\
 \therefore P(A|B) &= \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.2} & (5) \\
 &= \frac{1}{2}
 \end{aligned}$$

25

$$\begin{aligned}
 9. \quad \bar{x} &= 5 \text{ and } s_x = 2 \\
 (i) \quad y_i &\in \{12, 13, 14, 15, 16, 17, 18\} \\
 \text{Let } y_i &= x_i + 10. \\
 \text{Here } x_i &\in \{2, 3, 4, 5, 6, 7, 8\} \\
 \therefore \bar{y} &= \bar{x} + 10 = 5 + 10 = 15 & (5) \\
 \text{and } s_y &= s_x = 2
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad y_i &\in \{20, 30, 40, 50, 60, 70, 80\} \\
 \text{Let } y_i &= 10x_i \\
 \text{Here } x_i &\in \{2, 3, 4, 5, 6, 7, 8\} \\
 \therefore \bar{y} &= 10\bar{x} & (5) \\
 &= 10 \times 5 = 50 \\
 \text{and } s_y &= 10s_x = 10 \times 2 = 20 & (5)
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad \text{Let } y_i &= ax_i + b. & (5) \\
 \text{Then } \bar{y} &= a\bar{x} + b = 5a + b \\
 s_y^2 &= a^2 s_x^2 \\
 s_y &= |a| s_x & (5) \\
 &= 2|a|
 \end{aligned}$$

25

10.

u_i	-3	-2	-1	0	1	2
f_i	5	10	25	30	20	10
$f_i u_i$	-15	-20	-25	0	20	20

(5)

$$\bar{u} = \frac{\sum f_i u_i}{\sum f_i} = -\frac{20}{100} = -\frac{1}{5}$$

$$u_i = \frac{x_i - 35}{a}$$

(5)

$$\therefore \bar{x} = a\bar{u} + 35$$

$$33 = -\frac{a}{5} + 35$$

(5)

$$a = 10$$

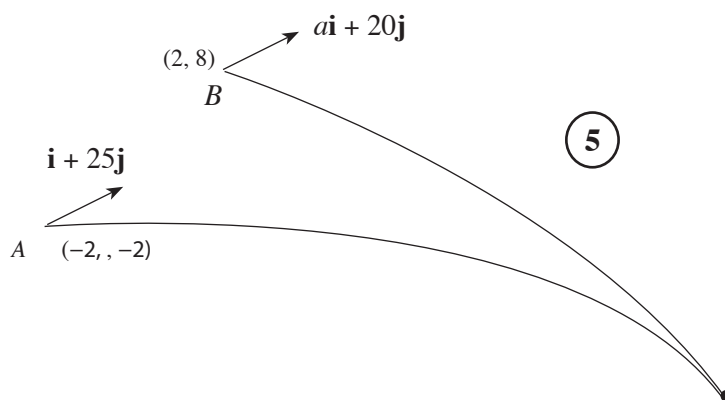
(5)

Intervals	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
f_i	5	10	25	30	20	10

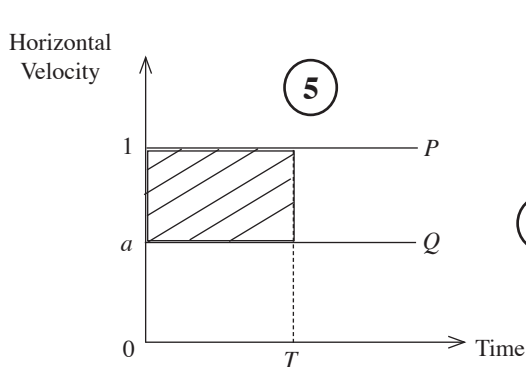
(5)

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11. (a)

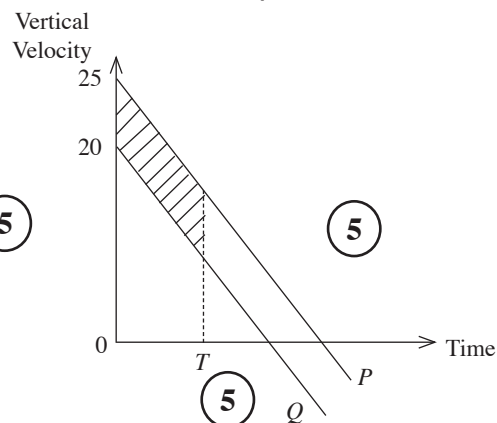


(5)



(5)

(5)



(5)

(5)

To collide,

$$\text{Vertical displacement of } P = \text{Vertical displacement of } Q + 10$$

$$\text{Vertical displacement of } P - \text{Vertical displacement of } Q = 10$$

(10)

$$\begin{aligned} 5T &= 10 \\ T &= 2 \end{aligned} \quad (5)$$

$$\begin{aligned} \text{Horizontal displacement of } P &= \text{Horizontal displacement of } Q + 4 \\ \text{Horizontal displacement of } P - \text{Horizontal displacement } Q &= 4 \end{aligned} \quad (10)$$

$$(1-a)2 = 4 \quad (5)$$

$$\begin{aligned} 1-a &= 2 \\ a &= -1 \end{aligned} \quad (5)$$

60

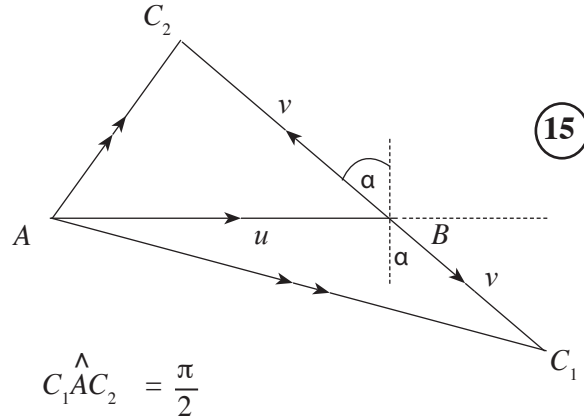
$$(b) \quad v(S, E) = \longrightarrow u$$

$$v(P, S) = \begin{array}{c} \nearrow \\ \searrow \end{array} \quad (5)$$

$$v(P, E) = v(P, S) + v(S, E)$$

$$\begin{array}{c} \text{Out} \\ \underline{v}(P, E) \end{array} = \begin{array}{c} \longrightarrow \\ \nearrow \\ \searrow \end{array} + \longrightarrow u \Rightarrow \overrightarrow{BC_1} + \overrightarrow{AB} = \overrightarrow{AC_1} \quad (5)$$

$$\begin{array}{c} \text{In} \\ \underline{v}(P, E) \end{array} = \begin{array}{c} \longrightarrow \\ \nearrow \\ \searrow \end{array} + \longrightarrow u \Rightarrow \overrightarrow{BC_2} + \overrightarrow{AB} = \overrightarrow{AC_2} \quad (5)$$



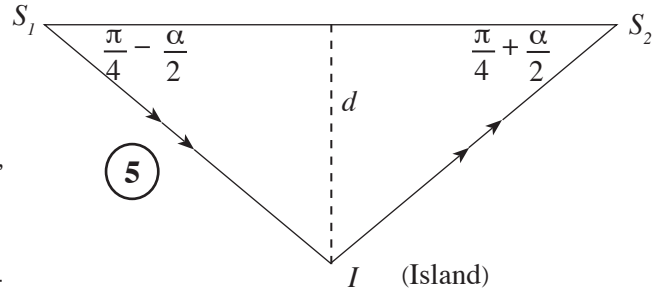
Hence, the circle with diameter C_1C_2 passes through the point A.

Furthermore, since the midpoint of C_1C_2 is B, (10)

$$BC_1 = BC_2 = BA = u \quad (5)$$

$$v = u$$

$$\begin{aligned} \angle BAC_1 &= \frac{\pi}{4} - \frac{\alpha}{2} \quad \text{and} \quad \angle BAC_2 = \frac{\pi}{4} + \frac{\alpha}{2} \end{aligned} \quad (5) \quad (5)$$



If the total time for the journey is t ,

$$\begin{aligned}
 t &= \frac{S_1 I}{AC_1} + \frac{S_2 I}{AC_2} \\
 &= \frac{S_1 I \sin(\frac{\pi}{4} - \frac{\alpha}{2})}{AC_1 \sin(\frac{\pi}{4} - \frac{\alpha}{2})} + \frac{S_2 I \sin(\frac{\pi}{4} + \frac{\alpha}{2})}{AC_2 \sin(\frac{\pi}{4} + \frac{\alpha}{2})} \\
 &= \frac{d}{v \cos \alpha} + \frac{d}{v \cos \alpha} = \frac{2d}{u \cos \alpha} \quad (\because v = u)
 \end{aligned}$$

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12. (a) By the law of conservation of energy;

$$\frac{1}{2}mu^2 - mga = mga \cos \theta + \frac{1}{2}mv^2 \quad (15)$$

$$v^2 - u^2 + 2ga(1 + \cos \theta) = 0 \quad (1) \quad (5)$$

Applying $F = ma$ to m

$$R + mg \cos \theta = \frac{mv^2}{a} \quad (2) \quad (10)$$

By substituting from (1) into (2)

$$R + mg \cos \theta = \frac{m}{a} [u^2 - 2ga(1 + \cos \theta)] \quad (5)$$

$$R = \frac{mu^2}{a} - mg(2 + 3 \cos \theta)$$

If the particle leaves the surface when OA makes an angle of α with the upward vertical, then at that point $R = 0$. (5)

$$\therefore u^2 - 2ga - 3ga \cos \alpha = 0$$

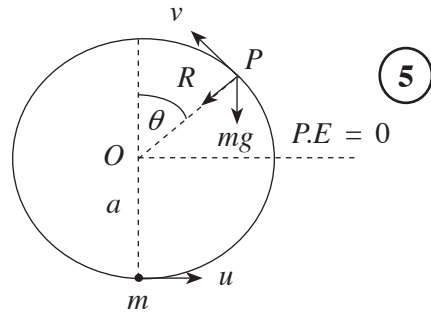
$$\cos \alpha = \frac{u^2 - 2ga}{3ga} > 0 \quad (\because u^2 > 2ga) \quad (5)$$

$\therefore \alpha$ is an acute angle.

Furthermore, since $0 < \alpha < \frac{\pi}{2}$, we have that $0 < \cos \alpha < 1$.

$$\frac{u^2 - 2ga}{3ga} < 1 \quad (5)$$

$$u^2 < 5ga$$



(5)

P.E = 0

(5)

(5)

(5)

(5)

(5)

When the particle of mass m leaves the surface, $\cos \alpha = \frac{1}{\sqrt{3}}$.

$$\frac{1}{\sqrt{3}} = \frac{u^2 - 2ga}{3ga}$$

$$u^2 - 2ga = \sqrt{3} ga$$

$$u^2 = (2 + \sqrt{3}) ga$$

(5)

$$\text{Then the velocity } v^2 = u^2 - 2ga \left(1 + \frac{1}{\sqrt{3}}\right) = 2ga + \sqrt{3} ga - 2ga - \frac{2ga}{\sqrt{3}} = \frac{ga}{\sqrt{3}}$$

(5)

After the particle of mass m leaves the surface of the sphere, its motion is that of a projectile.

In the ensuing motion, if the time taken to travel a horizontal distance $a \sin \alpha$ is t_0 ,

$$a \sin \alpha = (v \cos \alpha) t_0$$

(5)

$$\text{Then the distance traveled upward } y = (v \sin \alpha) t_0 - \frac{1}{2} g t_0^2$$

$$y = \frac{v \sin \alpha \times a \sin \alpha}{v \cos \alpha} - \frac{1}{2} \frac{ga^2 \sin^2 \alpha}{v^2 \cos^2 \alpha}$$

(5)

$$= \frac{\frac{2}{3} a}{\frac{1}{\sqrt{3}}} - \frac{ga^2}{\sqrt{3}} \frac{\frac{2}{3}}{\frac{1}{3}}$$

$$= \frac{2a}{\sqrt{3}} - \sqrt{3} a$$

$$= -\frac{a}{\sqrt{3}}$$

$$= -a \cos \alpha$$

(5)

Since the particle of mass m has travelled a distance of $a \cos \alpha$ downward when it passes the vertical line through O , it passes through the centre O of the sphere.

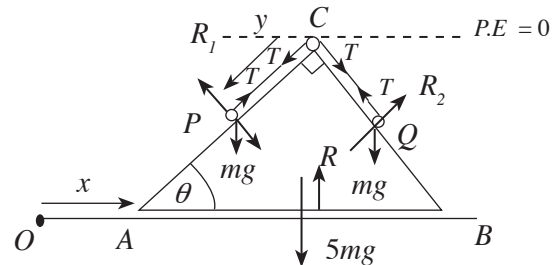
80

(b)

$$\mathbf{v}(P, O) = \begin{matrix} \dot{y} \\ \theta \\ \dot{x} \end{matrix}$$

$$\mathbf{v}(Q, O) = \begin{matrix} +\dot{y} \\ \frac{\pi}{2} - \theta \\ -2 \\ \dot{x} \end{matrix}$$

(5)



Applying the law of conservation of momentum;

$$\longrightarrow 5m \dot{x} + m(\dot{x} - \dot{y} \cos \theta) + m(\dot{x} - \dot{y} \sin \theta) = 0$$

(10)

$$7m \dot{x} = m \dot{y} (\cos \theta + \sin \theta)$$

$$7 \dot{x} = \dot{y} \left(\frac{3}{5} + \frac{4}{5} \right) \quad (5)$$

$$5 \dot{x} = \dot{y} \quad \text{—————} \quad (1)$$

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Applying the law of conservation of energy;

$$\frac{1}{2} 5m \dot{x}^2 + \frac{1}{2} m \{(\dot{x} - \dot{y} \cos \theta)^2 + (\dot{y} \sin \theta)^2\} \quad (20)$$

$$+ \frac{1}{2} m \{(\dot{x} - \dot{y} \sin \theta)^2 + (\dot{y} \cos \theta)^2\} - mgy \sin \theta - mg(l - y) \cos \theta = \text{constant}$$

$$5 \dot{x}^2 + \{ \dot{x}^2 + \dot{y}^2 - 2 \dot{x} \dot{y} \cos \theta \} + \{ \dot{x}^2 + \dot{y}^2 - 2 \dot{x} \dot{y} \sin \theta \}$$

$$- 2gy \sin \theta + 2gy \cos \theta = \text{constant} \quad (5)$$

$$7 \dot{x}^2 + 2 \dot{y}^2 - 2 \dot{x} \dot{y} \left(\frac{4}{5} + \frac{3}{5} \right) - 2gy \frac{4}{5} + 2gy \frac{3}{5} = \text{constant}$$

$$35 \dot{x}^2 + 10 \dot{y}^2 - 14 \dot{x} \dot{y} - 2gy = \text{constant} \quad \text{—————} \quad (2)$$

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From (1) and (2)

$$35 \dot{x}^2 + 250 \dot{x}^2 - 70 \dot{x}^2 - 2gy = \text{constant}$$

$$215 \dot{x}^2 - 2gy = \text{constant}$$

Differentiating with respect to t ,

$$430 \dot{x} \cdot \ddot{x} - 2g\dot{y} = 0 \quad (5)$$

$$430 \dot{x} \cdot \ddot{x} - 2g \cdot 5\dot{x} = 0 \quad (\because \dot{x} \neq 0) \quad (5)$$

$$\therefore \ddot{x} = \frac{g}{43}$$

10

Applying $F = ma$ to P

$$\text{Diagram: A block on an inclined plane at angle } \theta. \quad \therefore mg \sin \theta - T = m(\ddot{y} - \ddot{x} \cos \theta) \quad (5)$$

$$T = mg \sin \theta - m(5\ddot{x} - \ddot{x} \cos \theta) \quad (5)$$

$$= mg \frac{4}{5} - m\ddot{x} \left(5 - \frac{3}{5} \right)$$

$$= \frac{4mg}{5} - m \cdot \frac{1}{43} g \cdot \frac{22}{5}$$

$$= \frac{2mg}{5} \left\{ 2 - \frac{11}{43} \right\}$$

$$= \frac{2mg}{5} \times \frac{75}{43}$$

$$= \frac{30mg}{43} \quad (5)$$

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13. If the tension in the string when it is extended a length x from its natural length is T , then

$$T = \frac{\lambda x}{a} = \frac{2mgx}{a} \quad (5)$$

Applying $F = ma$ to the motion of the particle;

$$mg \sin 30^\circ - T = m\ddot{x} \quad (10)$$

$$mg \times \frac{1}{2} - \frac{2mgx}{a} = m\ddot{x}$$

$$\ddot{x} = -\frac{2g}{a} \left(x - \frac{a}{4}\right) \quad (1) \quad (5)$$

$$x = \frac{a}{4} + A \cot \omega t + B \sin \omega t \quad (2)$$

$$\dot{x} = -A\omega \sin \omega t + B\omega \cos \omega t \quad (3) \quad (5)$$

$$\ddot{x} = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t \quad (4) \quad (5)$$

$$= -\omega^2 (A \cos \omega t + B \sin \omega t)$$

$$\ddot{x} = -\omega^2 \left(x - \frac{a}{4}\right) \quad (5) \text{ by } (2) \quad (5)$$

By considering (1) and (5) $\omega^2 = \frac{2g}{a}$. (5)

$$t = 0 \text{ when } \dot{x} = 0. \quad (5) \quad \omega = \sqrt{\frac{2g}{a}}$$

From (3), $0 = B\omega$

Since $\omega \neq 0$ we have that $B = 0$. (5)

$x = a$ when $t = 0$ (5)

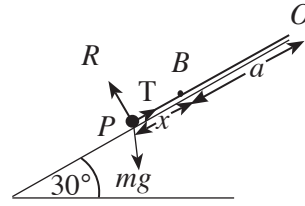
From (2), $a - \frac{a}{4} = A \Rightarrow A = \frac{3a}{4}$ (5)

$$\therefore x = \frac{3a}{4} \cos \omega t + \frac{a}{4}$$

$$x - \frac{a}{4} = \frac{3a}{4} \cos \omega t$$

The centre of the oscillation is given by $x - \frac{a}{4} = 0$. (5)

That is, $x = \frac{a}{4}$ is the centre. (5)



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When $\dot{x} = 0$ at the amplitude, let $t = t_1$.

$$0 = -A\omega \sin \omega t_1 \quad (5)$$

$$\sin \omega t_1 = 0$$

$$\omega t_1 = n\pi ; n \in \mathbb{Z}_0^+ \quad (5)$$

$$x - \frac{a}{4} = \frac{3a}{4} \cos \omega t_1$$

$$x - \frac{a}{4} = \pm \frac{3a}{4} \quad (5)$$

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$$\therefore \text{ the amplitude of the simple harmonic motion of the particle } = \frac{3a}{4} \quad (5)$$

Let the velocity of the particle when it first arrives at the natural length of the string be V .

$$\text{Then } x = 0. \quad (5)$$

$$\frac{3a}{4} \cos \omega t = -\frac{a}{4}$$

$$\cos \omega t = -\frac{1}{3} \quad (5)$$

$$\begin{aligned} V &= -A\omega \sin \omega t \\ &= -\frac{3a}{4} \sqrt{\frac{2g}{a}} \cdot \sqrt{1 - \cos^2 \omega t} \\ &= -\frac{3a}{4} \sqrt{\frac{2g}{a}} \sqrt{\frac{8}{9}} \quad (5) = -\frac{3a}{4} \sqrt{\frac{2g}{a}} \cdot \frac{2\sqrt{2}}{3} \\ &= -\sqrt{ag} \end{aligned}$$

$$\text{The velocity of the particle when } x = 0 \text{ is } \uparrow \sqrt{ag}. \quad (5)$$

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Let t_0 be the time when the particle first arrives at the natural length of the string.

$$\text{Then } x = 0. \quad (5)$$

$$-\frac{a}{4} = \frac{3a}{4} \cos \omega t_0$$

$$\cos \omega t_0 = -\frac{1}{3} \quad (5)$$

$$\omega t_0 = \pi - \cos^{-1} \left(\frac{1}{3} \right)$$

$$t_0 = \frac{1}{\omega} \left[\pi - \cos^{-1} \left(\frac{1}{3} \right) \right] = \sqrt{\frac{a}{2g}} \left[\pi - \cos^{-1} \left(\frac{1}{3} \right) \right] \quad (5)$$

The particle travels up to the point O under gravity.

If the time taken for the particle to travel from B to O is t_2 ,

$$S = ut + \frac{1}{2} at^2 \quad \uparrow$$

$$S = a, u = \sqrt{ag}, a = -g \sin 30^\circ$$

$$a = \sqrt{ag}t_2 - \frac{1}{2} \frac{g}{2} t_2^2 \quad (5)$$

$$\frac{g}{4} t_2^2 - \sqrt{ag} t_2 + a = 0$$

$$t_2 = \frac{\sqrt{ag} \pm \sqrt{ag - 4 \frac{g}{4} a}}{\frac{g}{2}}$$

$$t_2 = 2\sqrt{\frac{a}{g}} \quad (5)$$

$\therefore$ the time taken to travel to $O = t_0 + t_2$,

$$= \sqrt{\frac{a}{2g}} (\pi - \cos^{-1}(\frac{1}{3})) + 2\sqrt{\frac{a}{g}} \quad (5)$$

$$= \sqrt{\frac{a}{2g}} \left[\pi - \cos^{-1}(\frac{1}{3}) + 2\sqrt{2} \right]$$

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The string is at its maximum length at A ; that is when $x = a$,

$$T_A = \frac{\lambda a}{a} \quad (5)$$

$$= \lambda$$

$$T_A = 2mg \quad (5)$$

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$$14. (a) (i) |\underline{a}| = |\underline{b}| = |\underline{c}| = 1 \quad (5)$$

$$\text{If } (\underline{a} + 2\underline{b}) \perp (5\underline{a} - 4\underline{b}),$$

$$(\underline{a} + 2\underline{b}) \cdot (5\underline{a} - 4\underline{b}) = 0 \quad (5)$$

$$5\underline{a} \cdot \underline{a} + 10\underline{b} \cdot \underline{a} - 4\underline{a} \cdot \underline{b} - 8\underline{b} \cdot \underline{b} = 0$$

$$5|\underline{a}|^2 + 10\underline{a} \cdot \underline{b} - 4\underline{a} \cdot \underline{b} - 8|\underline{b}|^2 = 0$$

$$5 + 6\underline{a} \cdot \underline{b} - 8 = 0$$

$$6\underline{a} \cdot \underline{b} = 3 \quad (5)$$

$$\underline{a} \cdot \underline{b} = \frac{1}{2}$$

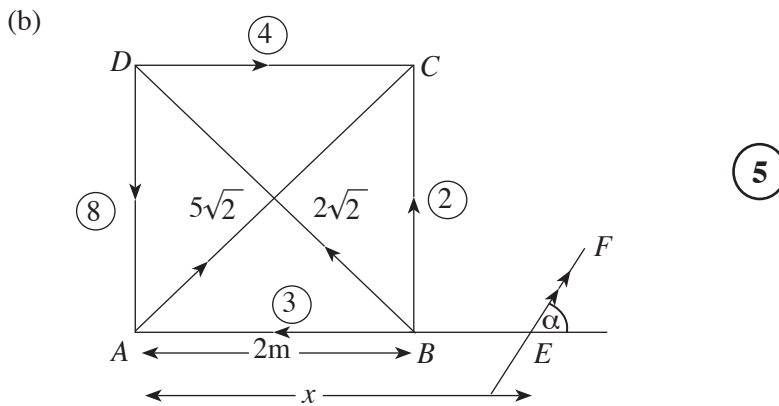
$$|\underline{a}| |\underline{b}| \cos \theta = \frac{1}{2} \quad (5)$$

$$1 \times 1 \cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ \quad (5)$$

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(ii) $|a - b|^2 + |b - c|^2 + |c - a|^2$
 $= (\underline{a} - \underline{b}) \cdot (\underline{a} - \underline{b}) + (\underline{b} - \underline{c}) \cdot (\underline{b} - \underline{c}) + (\underline{c} - \underline{a}) \cdot (\underline{c} - \underline{a})$ (5)
 $= |\underline{a}|^2 + |\underline{b}|^2 - 2\underline{a} \cdot \underline{b} + |\underline{b}|^2 + |\underline{c}|^2 - 2\underline{b} \cdot \underline{c} + |\underline{c}|^2 + |\underline{a}|^2 - 2\underline{c} \cdot \underline{a}$ (5)
 $= 6 - 2(\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{a})$ (5)
 $\therefore 2(\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{a}) = 6 - (|\underline{a} - \underline{b}|^2 + |\underline{b} - \underline{c}|^2 + |\underline{c} - \underline{a}|^2)$ (5) (1)
 $|\underline{a} + \underline{b} + \underline{c}|^2 \geq 0$ (5)
 $\therefore (\underline{a} + \underline{b} + \underline{c}) \cdot (\underline{a} + \underline{b} + \underline{c}) \geq 0$ (5)
 $|\underline{a}|^2 + \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{a} + |\underline{b}|^2 + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{a} + \underline{c} \cdot \underline{b} + |\underline{c}|^2 \geq 0$ (5)
 $3 + 2(\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{a}) \geq 0$ (2) (5)
 From (1) and (2)
 $3 + 6 - (|\underline{a} - \underline{b}|^2 + |\underline{b} - \underline{c}|^2 + |\underline{c} - \underline{a}|^2) \geq 0$ (5)
 $\therefore |\underline{a} - \underline{b}|^2 + |\underline{b} - \underline{c}|^2 + |\underline{c} - \underline{a}|^2 \leq 9$ (5)

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(i) $\vec{X} = 4 - 3 + 5\sqrt{2} \cos 45^\circ - 2\sqrt{2} \cos 45^\circ$ (5)
 $= 4\text{N}$
 $\uparrow Y = 2 - 8 + 5\sqrt{2} \cos 45^\circ + 2\sqrt{2} \cos 45^\circ$ (5)
 $= 1\text{N}$
 If the resultant is R ,
 $R = \sqrt{X^2 + Y^2} = \sqrt{4^2 + 1^2}$
 $= \sqrt{17}\text{N}$ (5)
 If the resultant makes an angle α with the horizontal,
 $\tan \alpha = \frac{1}{4}$
 $\alpha = \tan^{-1}\left(\frac{1}{4}\right)$ (5)

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If the point at which the line of action of the resultant intersects AB is E , let $AE = x$.

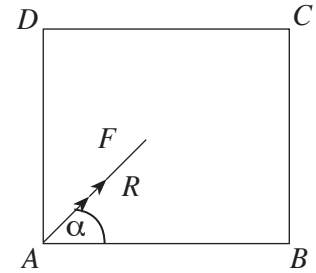
$$\curvearrowleft A \quad 1 \times x = 2 \times 2 - 4 \times 2 + 2\sqrt{2} \cdot 2 \cos 45^\circ \quad (5)$$

$$x = 0 \quad (5)$$

$A \equiv E$ (Coincides)

The line of action of the resultant passes through A . (5)

$\therefore$ for the system of forces to be in equilibrium, a force of $\sqrt{17}$ N should be introduced at A in the direction $\vec{FA}$. (5)



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(ii) To reduce the system to a couple of magnitude 39 Nm acting in the sense ABC , (5)

a force of $\sqrt{17}$ N should be introduced at a distance, say y , from A along

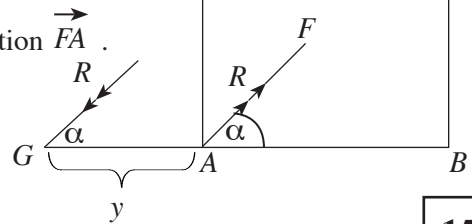
BA produced, such that it is parallel to AF and in the direction $\vec{FA}$.

$$\curvearrowleft ABC \quad \sqrt{17} \times AG \sin \alpha = 39 \quad (5)$$

$$\sqrt{17} \times AG \times \frac{1}{\sqrt{17}} = 39$$

$$AG = 39 \text{ m}$$

(5)



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(iii) The couple that should be introduced to reduce the system to a single force acting at B ; (5)

$$= \curvearrowleft \sqrt{17} \times BA \sin \alpha \quad (5)$$

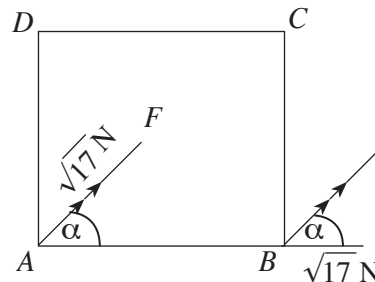
$$= \curvearrowleft \sqrt{17} \times 2 \times \frac{1}{\sqrt{17}} = 2 \text{ Nm} \quad (5)$$

Aliter

If the couple that should be introduced is M ,

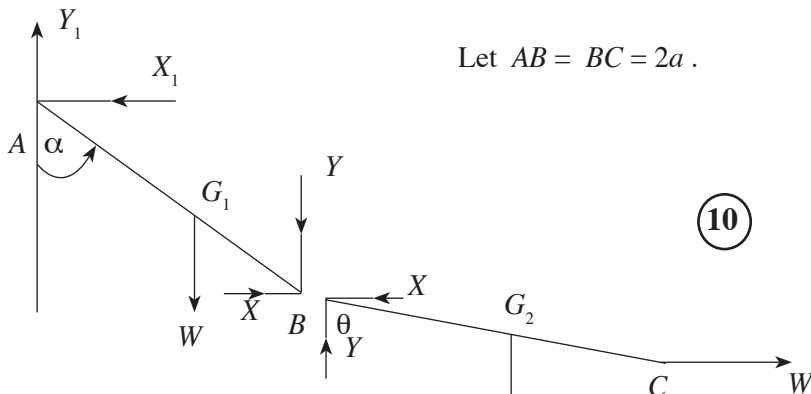
$$M - \sqrt{17} \times 2 \sin \alpha = 0$$

$$M = \sqrt{17} \times 2 \times \frac{1}{\sqrt{17}} = 2 \text{ Nm}$$



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15. (a)



Let $AB = BC = 2a$.

(10)

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(i) Taking moments about B for BC (5)

$$W a \sin \theta = 2 W a \cos \theta$$

$$\tan \theta = 2$$

(5)

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Considering the equilibrium of BC ,

$$\leftarrow X = W \quad (5)$$

$$\uparrow Y = W \quad (5)$$

$$\therefore R_B = \sqrt{W^2 + W^2} \quad (5)$$

$$= \sqrt{2} W$$

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R_B makes an angle of $\tan^{-1} 1 = \frac{\pi}{4}$ with the horizontal (5)

$AB \odot \curvearrowright A$

$$X \cdot 2 \cos \alpha = W \sin \alpha + Y \cdot 2 \sin \alpha \quad (10)$$

$$W \cdot 2 \cos \alpha = W \sin \alpha + W \cdot 2 \sin \alpha$$

$$\frac{2}{3} = \tan \alpha$$

$$\alpha = \tan^{-1} \left(\frac{2}{3} \right) \quad (5)$$

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By considering the equilibrium of AB ,

$$\rightarrow X_1 = X = W \quad (5)$$

$$\uparrow Y_1 = 2W \quad (5)$$

$$\therefore R_A = \sqrt{X_1^2 + Y_1^2}$$

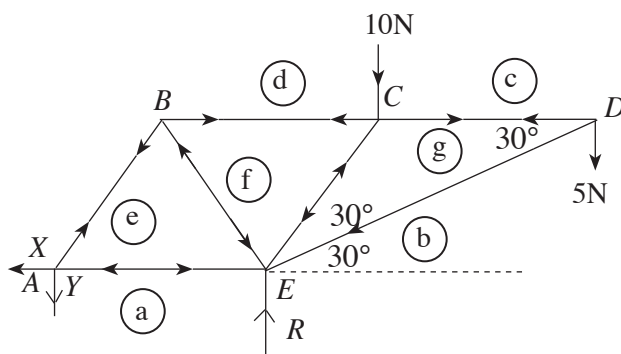
$$= \sqrt{5} W \quad (5)$$

R_A makes an angle of $\tan^{-1}(2)$ with the horizontal

(5)

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(b)



Let the length of a rod (other than DE) be $2a$.

(i) Considering the system, $\curvearrowright A$

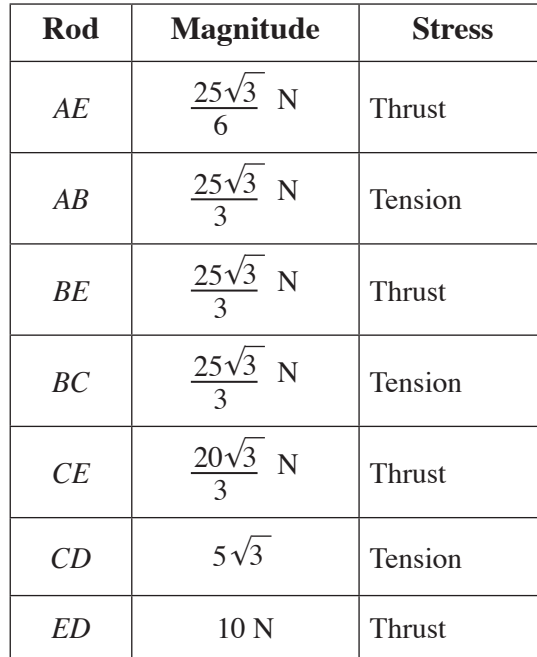
$$R \cdot 2a - 10 \times 3a - 5 \times 5a = 0 \quad (5)$$

$$R = \frac{55}{2} \text{ N} \quad (5)$$

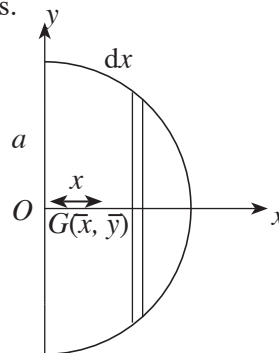
$$\therefore \text{the vertical force acting at } E = \frac{55}{2} \text{ N}$$

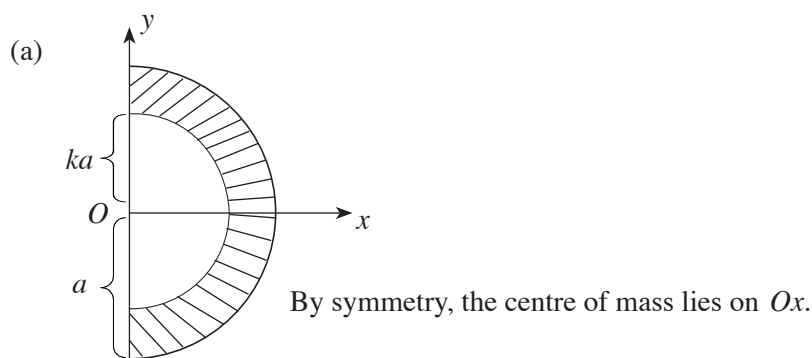
The vertical component of the reaction at the hinge A = $\downarrow \frac{25}{2}$ N (5)

Horizontal component = 0 (5)



Rod	Magnitude	Stress
AE	$\frac{25\sqrt{3}}{6}$ N	Thrust
AB	$\frac{25\sqrt{3}}{3}$ N	Tension
BE	$\frac{25\sqrt{3}}{3}$ N	Thrust
BC	$\frac{25\sqrt{3}}{3}$ N	Tension
CE	$\frac{20\sqrt{3}}{3}$ N	Thrust
CD	$5\sqrt{3}$	Tension
ED	10 N	Thrust

$$\begin{aligned}\bar{y} &= 0 && \textcircled{5} \\ \bar{x} &= \frac{\int_0^a \pi \rho x (a^2 - x^2) \, dx}{\int_0^a \pi \rho (a^2 - x^2) \, dx} && \textcircled{10} \\ &= \frac{\left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a}{\left[a^2 x - \frac{x^3}{3} \right]_0^a} && \textcircled{5} \\ &= \frac{3a}{8} && \textcircled{5} \\ \therefore G &\equiv \left(\frac{3a}{8}, 0 \right)\end{aligned}$$




Object	Mass	Distance from O to the centre of mass
Hemisphere	$\frac{2}{3} \pi a^3 \rho$	$\frac{3a}{8}$
Hemisphere which is removed	$\frac{2}{3} \pi (ka)^3 \rho$	$\frac{3ka}{8}$
Remaining portion	$\frac{2}{3} \pi a^3 \rho (1 - k^3)$	$\bar{x}$

(10)

(5)

$$\bar{x} = \frac{\frac{2}{3} \pi a^3 \rho \frac{3a}{8} - \frac{2}{3} \pi k^3 a^3 \rho \frac{3ka}{8}}{\frac{2}{3} \pi a^3 \rho (1 - k^3)} \quad (15)$$

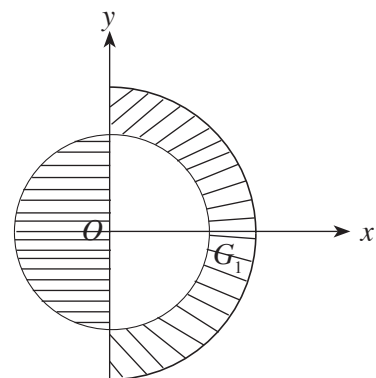
$$= \frac{\frac{3a}{8} (1 - k^4)}{(1 - k^3)} = \frac{3a (1 + k^2) (1 - k) (1 + k)}{8 (1 - k) (1 + k + k^2)}$$

$$= \frac{3a (1 + k^2) (1 + k)}{8 (1 + k + k^2)} \quad (10)$$

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(b) Let the centre of mass be $G_1(\bar{x}_1, \bar{y}_1)$

When the hemisphere that is removed is attached to the remaining portion as shown in the figure, due to symmetry about Ox we have $\bar{y}_1 = 0$. (5)



(i) Let the mass of the hemisphere that was removed be m , and the mass of the hemisphere of radius a be M .

$$\frac{m}{M} = \frac{\frac{2\pi}{3} k^3 a^3 \rho}{\frac{2\pi}{3} a^3 \rho} = k^3 \quad (5)$$

$$m = Mk^3$$

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- (ii) The distance from O to the centre of mass of the composite object is $\bar{x}_1$.

$$\bar{x}_1 = \frac{(M-m)\bar{x} + m\left(-\frac{3}{8}ka\right)}{(M-m) + m} \quad (15)$$

Since $(M-m)\bar{x} = M\left(\frac{3a}{8}\right) - m\left(\frac{3ka}{8}\right)$, (5)

$$\begin{aligned} \bar{x}_1 &= \frac{M\left(\frac{3a}{8}\right) - m\left(\frac{3ka}{8}\right) - m\left(\frac{3ka}{8}\right)}{M} \\ &= \frac{\frac{3a}{8} (M - 2mk)}{M} \\ &= \frac{3a}{8} \left(1 - \frac{2m}{M}k\right) \quad (10) \end{aligned}$$

The distance from O to the centre of mass of the composite object $= \frac{3a}{8} (1 - 2k^4)$

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- (iii) The centre of mass G_1 should coincide

(5)

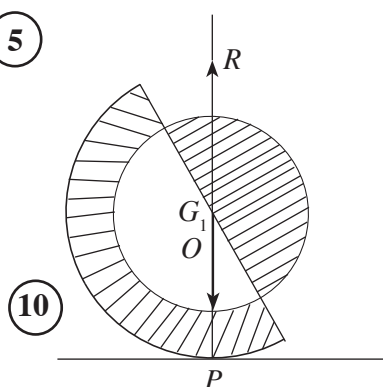
with O . That is, $\bar{x}_1 = 0$. (5)

$$\frac{3a}{8} (1 - 2k^4) = 0 \quad (5)$$

$$2k^4 = 1$$

$$k^2 = \pm \frac{1}{\sqrt{2}} \quad (5)$$

$$\text{Since } k^2 > 0, k^2 = \frac{1}{\sqrt{2}}$$



(10)

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17. (a) $P(A) = 0.1$, $P(A \cup B) = 0.37$ and $P(C) = 0.2$

(i) $P(A \cup B) = P(A) + P(B) - P(A)P(B)$ (5)

(Since A and B are independent)

$$0.37 = 0.1 + P(B) - 0.1P(B)$$

(5)

$$0.37 - 0.1 = 0.9P(B)$$

$$0.3 = P(B)$$

(5)

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(ii) $P(B' | A') = \frac{P(B' \cap A')}{P(A')}$ (5)

Here $P(B' \cap A') = P[(B \cup A)'] = 1 - P(A \cup B)$ (5)

$$= 1 - 0.37 = 0.63$$

$$P(A') = 1 - P(A) = 1 - 0.1 = 0.9$$

(5)

$$\therefore P(B' | A') = \frac{0.63}{0.9} = 0.7$$

(5)

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$$\begin{aligned}
 \text{(iii) } P(A' \cap B' \cap C) &= P(A') P(B') P(C) && \textcircled{5} \\
 &= 0.9 \times 0.7 \times 0.2 \\
 &= 0.126 && \textcircled{5}
 \end{aligned}$$

10

$$\text{(iv) } X : (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \quad \textcircled{5}$$

$$\begin{aligned}
 P(X) &= P(A \cap B' \cap C') + P(A' \cap B \cap C') + P(A' \cap B' \cap C) \\
 &= P(A) P(B') P(C') + P(A') P(B) P(C') + P(A') P(B') P(C) && \textcircled{5} \\
 &= 0.1 \times 0.7 \times 0.8 + 0.9 \times 0.3 \times 0.8 + 0.9 \times 0.7 \times 0.2 && \textcircled{10} \\
 &= 0.398
 \end{aligned}$$

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$$\begin{aligned}
 \Rightarrow P(A | X) &= \frac{P(A \cap X)}{P(X)} \\
 &= \frac{P(A \cap B' \cap C')}{P(X)} && \textcircled{5} \\
 &= \frac{0.1 \times 0.7 \times 0.8}{0.398} \\
 &= \frac{56}{398} && \textcircled{5} \\
 &= \frac{28}{199}
 \end{aligned}$$

10

$$\text{(b) (i) } (\alpha) \text{ Mean } 8 = \frac{\sum_{r=1}^n x_r}{n} \quad \textcircled{5}$$

$$\begin{aligned}
 \text{Mean value of the marks } \bar{x} &= \frac{28 + 56 + 23 + 94 + 8 + 5 + 13 + 846}{28} && \textcircled{5} \\
 &= \frac{1073}{28} \\
 &= 38.32 && \textcircled{5}
 \end{aligned}$$

15

(β) Since the number 94 should have been 49;

Thus the drop of mark = 45

Since the number 05 should have been 50; \textcircled{5}

Thus the rise of mark = 45

Therefore no change in the mean value. \textcircled{5}

10

$$\text{Standard deviation} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}} \quad (5)$$

$$\begin{aligned} \text{Variance } s_x^2 &= \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n} \\ &= \frac{\sum_{i=1}^n (x_i^2 - 2x_i\bar{x} + \bar{x}^2)}{n} \quad (5) \\ &= \frac{\sum_{i=1}^n x_i^2}{n} - 2\bar{x} \frac{\sum_{i=1}^n x_i}{n} + \bar{x}^2 \\ &= \frac{\sum_{i=1}^n x_i^2}{n} - 2\bar{x}^2 + \bar{x}^2 \quad (5) \\ &= \frac{\sum_{i=1}^n x_i^2}{n} - \bar{x}^2 \end{aligned}$$

Let $X = \{x_1, x_2, \dots, x_{20}\}$ and $Y = \{y_1, y_2, \dots, y_{10}\}$.

$$\text{Since } \sum_{i=1}^{20} x_i = 320, \quad \sum_{i=1}^{20} x_i^2 = 5840 \quad (5)$$

and

$$\sum_{i=1}^{10} y_i = 130, \quad \sum_{i=1}^{10} y_i^2 = 2380 \quad (5)$$

$$\therefore \bar{x} = \frac{\sum_{i=1}^{20} x_i}{20} = \frac{320}{20} = 16 \quad (5)$$

$$\text{and } s_x^2 = \frac{\sum_{i=1}^{20} x_i^2}{20} - 16^2 = \frac{5840}{20} - 16^2$$

$$= 292 - 256 = 36$$

$$\therefore s_x = 6 \quad (5)$$

$$\bar{y} = \frac{\sum_{i=1}^{10} y_i}{10} = \frac{130}{10} = 13 \quad (5)$$

$$s_y^2 = \frac{\sum_{i=1}^{10} y_i^2}{10} - 13^2 = \frac{2380}{10} - 169 = 69$$

$$\therefore s_y = 8.30 \quad (5)$$

Let $Z = X \cup Y$.

$$\begin{aligned} \bar{z} &= \frac{\sum_{i=1}^{20} x_i + \sum_{i=1}^{10} y_i}{30} \\ &= \frac{320 + 130}{30} = 15 \quad (5) \end{aligned}$$

$$s_z^2 = \frac{\sum_{i=1}^{20} x_i^2 + \sum_{i=1}^{10} y_i^2}{30} - \bar{z}^2 \quad (5)$$

$$= 274 - 225 = 49$$

$$s_z = 7 \quad (5)$$

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